CONCEPTUAL FOUNDATIONS OF QUANTUM FIELD THEORY

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and

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PREFACE

This book is about the physical interpretation of quantum field theory (QFT). It is not in any sense a textbook of the formalism of quantum field theory, as that is already dealt with at many levels of sophistication in the available literature. Since the book is intended to be read by philosophers as well as physicists, we have tried to make it self-contained, assuming no more than the elements of non-relativistic quantum mechanics (QM) of systems with a finite number of degrees of freedom, and showing without oppressive attention to mathematical rigour how this gets extended to relativistic quantum field theory.

The interpretation of QFT carries with it all of the problems traditionally associated with QM. These have been discussed at some length in a recent book entitled, 'Incompleteness, Nonlocality and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics', by one of the present authors (MR). In the present work we shall be concerned with the question of whether essentially new interpretative problems arise in the extension to QFT.

We shall be concerned, for example, with the problems about the identity and individuality of particles in QM and the relation with so-called field quanta. We shall examine how the problem of wave-particle duality looks from the perspective of QFT; we shall discuss the Feynman diagrams which give such graphic illustration to perturbative calculations in QFT; we shall be concerned with the connection between matter and force, and the whole question of unification in modern particle physics. We shall be at pains to stress the differences between interpreting interacting and non-interacting fields. There will be a discussion of renormalisation, running coupling constants, the renormalisation group, and some remarks about the special problems of gauge theory, quantizing the gravitational field, of supersymmetry and of string, and of quantum cosmology.

The book will have achieved its purpose if it removes common misconceptions about QFT among philosophers, and sets the physicists to thinking harder about what they are doing than some of them are used to.

We are grateful to colleagues, friends and former students (by no means non-intersecting sets), for debating these questions with us. We would particularly like to mention Ray Rivers, Ian Aitchison, Chris Isham, Gordon Fleming, Fritz Rohrlich, Sylvan Schweber, Bob Weingaard, Jeremy Butterfield, Steven French, Simon Saunders, Andrew Westwell-Roper and Tien-Yu Cao. One of us (MR) owes a debt of gratitude to Leonardo Castillejo for patient introduction to some of the subtleties and difficulties of the subject, when a raw graduate student, but sins both of commission and ommission are the sole responsibilities of the two authors.

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1. INTRODUCTION

Quantum field theory (QFT) has a history nearly as long as that of quantum theory itself. Indeed the quantum theory originated with the problem of the distribution of energy in the radiation from a black body. This problem is, in essence, concerned with the interaction between 'resonators' in the black body and the electromagnetic field. A convenient model of a black body is a cavity in which radiation, introduced through a small aperture in the wall of the cavity, is never able to escape with any measurable probability, provided the ratio of aperture to total surface area is made vanishingly small, i.e. such a cavity is a perfect absorber of radiation incident on the aperture. Planck tackled the problem by calculating the ratio of the mean energy density u per unit range of frequency ν of radiation to the mean energy U of the resonators or oscillators in the wall of the cavity using classical electromagnetic theory.

The result was,

$$u/U = \sim\!\!(8D/c^3)\nu^2 \quad 8\pi/c^3 \cdot \nu^2$$

Planck proceeded to calculate u from this formula, by first calculating U using the 'quantum' assumption that the total energy distributed among the oscillators of frequency ν was an integral multiple of a discrete unit of energy $h\nu$, where h was the famous Planck constant.

The final result was the Planck distribution law (1900):

$$u(\nu, T) = 8\pi\nu^2/c^3 \cdot \frac{h\nu}{e^{h\nu/kT} - 1}$$

where T is the absolute temperature and k is Boltzmann's constant.

But Debye, in 1910, following an earlier suggestion of Ehrenfest (1908), showed how one could calculate u directly by applying the quantum condition, not to the oscillators in the walls of the cavity, but to the normal modes of the electromagnetic field itself. This was the first example of the quantization of a field, considered as a 'mechanical' system with an infinite number of degrees of freedom, as opposed to the quantization of mechnical systems such as the harmonic oscillator with a finite number of degrees of freedom, so a 'discreteness' of quantization was now attributed directly to the electromagnetic field , instead of the classical continuous treatment envisaged in the original work of Planck.

But another way of introducing discreteness into the electromagnetic field had been introduced by Einstein in 1905 with his light-quantum hypothesis. This suggested that light was actually corpuscular in character; the light corpuscles or photons as they later came to be known, carried energy $h\nu$ and momentum $\frac{h\nu}{c}$, so the 'rest mass', defined by the relation

$$E^2 = p^2c^2 + m^2c^4$$

came out to be zero.

The connection between Debye's quantized modes of the field and Einstein's light-quantum was not at all clear. The mystery deepened when, in 1924, Bose showed how to derive Planck's radiation law by considering radiation as a 'gas' of Einsteinian light-quanta, subject to a novel statistical treatment that effectively treated the light-quanta as non-individuals. So there seemed to be two ways of treating electromagnetic radiation: either as a field, subject to quantization, or as a collection of particles, but not as ordinary classical particles, but particles of a quantal nature, bereft of individuality.

The two aspects of light, the field and the particle aspects constituted the problem of wave-particle duality. But the two approaches of calculating the Planck distribution law due to Debye and Bose seemed to show that the same problem could be tackled from either a field or a particle point of view, provided the quantal features were introduced into each account.

The quantized field and the quantized collection of particles were in some sense equivalent. The quantization of the continuous field endowed it with particle-like properties, while the quantization of particles endowed them with wave-like properties.

Einstein's light-quantum conception was not generally accepted by the physics community until the explanation afforded by it of the Compton effect (1923), in which the change in freqency (and hence energy) of a photon scattered off a free electron was computed from the classical conservation laws of energy and momentum in the collision between the two particles, namely the photon and the electron.

The fusion of wave and particle aspects of light was vividly demonstrated by Einstein's calculation (1909) of the energy fluctuations in cavity radiation to be expected on the basis of the Planck distribution law. This was exhibited as the sum of two terms, one characteristic of a collection of classical particles, the other of a classical wave field.

A similar demonstration of wave-particle duality for the case of material particles was provided by Einstein in the extension of Bose's method of calculating the energy distribution for a collection of photons, to the case of a quantum gas comprised of a fixed number of particles of non-vanishing rest mass. The energy fluctuation formula for this case again shows the characteristic sum of two contributions, one to be expected of a classical particle theory, the other of a classical wave theory.

In the meantime, wave properties of the electron had already been proposed by Louis de Broglie in his 1923 thesis and these predictions were brilliantly confirmed by the phenomenon of electron diffraction (Davisson & Germer (1927), and GP. Thomson (1927)).

De Broglie's ~~The~~ work led directly to Schrödinger's wave mechanics (1926) and at first Schrödinger tried to interpret his wave function realistically in terms of picturing the material electron as in fact an extended field-like charge cloud. This interpretation met with apparently insurmountable difficulties due to the dispersal of the charge clouds or wave packets with the passage of time, and also the problem of describing the many-particle system ~~problems~~ in terms of 3N-dimensional configuration spaces, rather than the ordinary physical three-dimensional space. Only for a single particle (N=1) could the two spaces be identified.

The Schrödinger interpretation of the wave function was replaced in 1926 by the Born interpretation in which the electrons were conceived of as particles (|ψ|² and |ψ|), the squared modulus of the wave function, measured the probability density of a final electron in the neighbourhood of the positional argument of the wave function. So with Born we were back to particles as the ultimate ontology, with a probabilistic interpretation of the locations.

The Born interpretation was given a much more general setting in the transformation theory of Jordan and Dirac, in which the elements of the transformation matrix were interpreted as probability amplitudes for finding the electron to exhibit on measurement the value of one observable, given that it had been prepared in the state in which some other observable had a sharp value. These probabilities governed the quantum ~~'s~~ 'jumps' to which Schrödinger had such a strong aversion.

But wave mechanics had already been shown in 1926, by Schrödinger himself, to be empirically equivalent to the other version of the new quantum mechanics, the so-called matrix mechanics of Heisenberg, Born and Jordan. This was a theory in which the electron is conceived of as a particle, but its observable properties, such as position and

momentum, were treated as non-commuting matrices, rather than the familiar real numbers of classical mechanics.

Matrix mechanics was developed in three famous papers, Heisenberg ~~(back in~~ (1925); Born and Heisenberg (1925); Born, Jordan and Heisenberg ~~(back in~~ (1926). In this last paper a preliminary attempt was made to extend the application of matrix mechanics from material systems to the electromagnetic field, following here very much the spirit ~~viewpoint~~ of Debye's work already referred to.

The really seminal paper on the quantum theory of the electromagnetic field was, however, contributed by Dirac (1927). In this paper Dirac discussed the quantization of the electromagnetic field introducing, as in the work of Debye, a discreteness into the possible energies possessed by the field. However, he also considered the radiation field as an assemblage of 'particles', ie, the photon gas model of Bose. Somewhat incoherently Dirac discussed the properties of ~~the~~ an assemblage of bosons governed by the non-relativistic Schrödinger equation and showed that the quantum mechanics of this assembly was, in fact, exactly equivalent to what ~~one~~ would obtain if the one-particle Schrödinger equation was regarded as a mechanical field subject to a canonical quantization procedure.

Now two points of view can be taken concerning the ontological implications of this second approach to the quantum theory of radiation. In the first place it could be regarded merely as a mathematical reformulation of ~~the~~ a particle theory, that is to say a theory of an assemblage of particles governed by the N-particle Schrödinger equation, but on the other hand it could also be regarded as a wave theory in which the one-particle Schrödinger equation ~~is~~ was regarded as a classical wave to be subjected to a canonical quantization just as the Maxwell equations for the electromagnetic field, and the particle-like behaviour was to be interpreted not in terms of genuine particles but merely of a discreteness in the attributes of the field which gave it a formal analogy to the properties of a collection of particles. The ~~second~~ approach to the quantum mechanics of indistinguishable particles we shall refer to a second quantization. The term 'second' refers to the fact that the one-particle Schrödinger equation is regarded as a result of first quantization and this is now to be re-interpreted as a classical theory to be subjected to a second quantization. But if we want to take this second quantization idea seriously from the ontological point of view, we shall prefer the term field quantization, to indicate that the ontology is ultimately one of fields rather than of particles. In the case of radiation we have correspondingly two views as to what the radiation process consisted in. On the first approach, that of field quantization of the Maxwell

and reserve the term 'second quantization' for the mathematical reformulation of the many-particle theory.

equations, radiation of a discrete quantized light-quantum consisted of the transition of the state of the field from one mode of excitation to another mode of excitation. On the other hand, on the second-quantized point of view, radiation consists in the actual creation of a new particle. Similarly, absorption of radiation consists in the destruction or annihilation of a particle of radiation.

Now Dirac himself, while being the first to introduce the second-quantized formulation of radiation theory, was unhappy with the idea that particles could actually be created or destroyed. He introduced the idea of a reservoir state, such that an infinite number of unobservable photons occupied this state, and radiation consisted in the transition from this reservoir state to some actual observable state of propagation of the particle. As we shall see in a moment, this idea of a reservoir state of photons is rather similar to Dirac's treatment of the hole theory of the positron.

The subject of second quantization was taken up by Jordan and Klein who extended the method to allow for arbitrary two-particle interactions between an assemblage of bosons, and was adapted to deal with the case of fermions in a classic paper by Wigner and Jordan in which they showed that the canonical quantization of the one-particle Schrödinger equation according to the replacement of Poisson brackets by commutator brackets had now to be substituted with a new scheme, in which the commutator brackets were replaced by anti-commutator brackets.

In the meantime, Jordan and Pauli had developed a relativisticly invariant formulation of the commutation properties of what we have termed the field approach, that is to say they investigated the commutation properties of field amplitudes at two distinct space-time points.

The development of a fully-fledged relativistic quantum electrodynamics was carried through in the period 1929 to 1930 by Heisenberg and Pauli. The one-particle Dirac equation was substituted for the non-relativistic Schrodinger equation in order to provide a field quantization for the electrons, while the Maxwell equations were quantized in the manner already indicated by Dirac.

At this point we digress briefly, to discuss the well-known difficulty of negative energy solutions associated with the one-particle Dirac equation. This would lead to the possibility of transitions from positive energy states into these negative energy states, ~~and mean~~ so ~~moreover~~ that electrons could not persevere in a state of positive energy. To deal with this problem, Dirac postulated an infinite sea of negative energy states filled, in accordance with the Pauli exclusion principle, by one electron occupying each such state. Only if a hole was created in this negative energy sea by lifting a negative energy electron into a positive energy state

could an electron make a possible transition in the reverse direction, from positive energy to negative energy. Although initially Dirac identified these negative energy holes with protons, it was immediately pointed out by Weyl that the masses of the hole and the original particle must in fact be the same. Furthermore, Oppenheimer demonstrated that if the holes were identified with protons, then electrons and protons should annihilate one another, the electron with positive energy falling into the negative energy proton hole at an unacceptably high rate. Accordingly it became clear that the negative energy holes must be identified with ~~the~~ electrons, but now with a positive charge. In fact these particles were discovered in cosmic rays by Anderson in 1932, and confirmed by the work of Blackett.

The formalism of second quantization was developed in a very neat way by Vladimir Fock in 1930. We shall discuss this work in Chapter Four below.

The first person to embrace wholeheartedly the possibility of genuine particle annihilation and creation was Enrico Fermi in 1933 with his theory of beta decay. This was modeled on the theory of electromagnetic radiation, ~~in which~~ electrons and antineutrinos ~~were~~ being radiated as a result of a transition between nucleon states. For some time there had been controversy as to whether the electrons observed to be emitted by radioactive nuclei ~~and~~ in beta decay existed before the decay in the nucleus. There were a number of convincing arguments that this could not be the case: as a result of the uncertainty of relations, these electrons would have unacceptably high energy and also would contrbute to the total spin of the nucleus giving different values to the total spin than those deduced from the properties of band spectra for molecules such as nitrogen. All these problems were neatly dealt with by Fermi by assuming that the electrons and antineutrinos were actually created in the process of beta decay. We shall discuss the philosophical repercussions of this revolutionary idea in Chapter Three.

The Dirac electron had spin ~~a~~ one half, and the problems associated with quantizing the Dirac field were those those presented by the fact that the energy density of the field was apparently not positive-definite, and this was ~~s~~ solved using the idea of positron-holes in the way we ~~ve~~ have described. The next major development in quantum field theory was the discussion ~~of~~ by Pauli and Weisskopf in 1934 of the quantum theory of the spin-zero field, governed by the Klein-Gordon equation. In this case, the energy was positive-definite, but problems were associated in the interpretation of the one-particle equation with the fact that the probability density was not positive-definite. This was now taken care of in the Pauli-Weisskopf treatment, by regarding the second-quantized Klein-Gordon equation as describing collections of positively and negatively charged spin-zero paricles and reinterpreting the probability density in terms of a current density for which a requirement of positive-definiteness was clearly not required.

During the 1930's and 1940's, quantum field theory was rapidly developed to deal with the increasing number of new particles which were being discovered in cosmic rays, and which were characteristically unstable, decaying in times of the order of 10^{-10} second. ~~indecay~~. The creation and annihilation of such particles was beautifully adapted to description by the formalism of quantum field theory. Perhaps most famously Yukawa in 1935 discussed the property ies of a scalar field which would mediate interactions between nuclear particles in the

same way that the electromagnetic field mediated interaction between electrically charged particles. The field quanta associated with Yukawa's field were later identified with the so-called 'mesotrons' which had been discovered ~~as part of~~ among the decay products of cosmic rays incident on the upper atmosphere. In fact these mesotrons turned out on later analysis to be ~~as~~ what we now know as muons, which are weakly interacting particles ~~with~~ and have nothing to do ~~connection~~ with the strong nuclear interactions. In fact it was~~n't~~ not until 1947 that Powell and his co-workers at Bristol discovered ~~a~~ the 'pion', which is the real candidate for the Yukawa field quantum.

In spite of the successes of quantum field theory in providing a framework for understanding the ephemeral nature of the elementary particles that were being discovered, a profound theoretical difficulty confronted the theory from its very inception. It was pointed out by Ehrenfest, and confirmed by detailed calculations of Waller and Oppenheimer, that the interaction of an electron with its own field, that is to say, ~~or~~ the absorption of a virtual photon emitted by such an electron by the very electron which emitted it, would lead to divergent, that is to say infinite, contributions to the self-energy of the electron. Furthermore it transpired that when any process involving the radiation of virtual photons was considered beyond the first non-vanishing order of perturbation theory, the self-energy effect produced an infinite result for the prediction of

the associated cross-sections. In other words, if a calculation were pursued with an attempt to increase the accuracy of the numerical predictions, the result passed from being in reasonable accord with experiment to being the absurd answer of an infinite prediction. During the 1930's, much effort was devoted to so-called 'subtraction physics', in which these infinite quantities were simply subtracted, i.e. ignored, in the course of calculation. But the properly and effective way of dealing with this problem was only developed after the second World War, during the period 1945-1947, using the technique of renormalization. This dates back to Dirac, and was also particularly advocated by Hendrick Kramers, but it was developed in its modern form by Schwinger, Tomonaga, Feynman, and Dyson. The idea was that the infinite quantity should be absorbed into the definition of the mass, and indeed the charge, of the electron, and these renormalized values should be identified with the experimental values actually observed in the laboratory. It was shown by Dyson that this program of renormalization could be carried out for all orders of perturbation theory, although this still left open the problem of the convergence of the resulting renormalized perturbation series.

The possibility of removing divergence in quantum field theory by renormalization became the guiding factor in distinguishing those field theories, which were of physical significance, from those which were regarded as having no clear physical interpretation. In particular, it emerged during the 1960's that a very important class of renormalizable theories was provided by the so-called 'gauge field' theories, of which quantum electrodynamics was a paradigmatic prototype. The ~~non-Abelian~~ gauge theories ~~involved~~ utilized in the Weinberg-Salam electroweak unified theory, involved a non-Abelian gauge group and led in turn to developments in grand unification, that attempted to combine the electromagnetic, weak and strong interactions in one all-embracing symmetry scheme. The background to these gauge theories will be discussed in Chapter 11 below.

A further very significant development was ~~to~~ the attempt to incorporate gravitation as the fourth fundamental force in the unified scheme. Very recently, in the theory of so-called supersymmetric strings, an example has been provided of a theory which is actually finite, and appears to make physical sense to all orders in perturbation theory. Supersymmetric string theory has produced great excitement among the physics community as a candidate for a theory of everything. We shall make some remarks on this ~~development~~ prospect in the chapter discussing gauge theories and their developments.

In Chapter 12, we shall discuss the question of unification and its ontological significance, with particular reference to the blurring of the distinction between matter and force, which is a notable feature of quantum field theories. In brief, the interaction between material particles, i.e. field quanta, ~~is~~ is mediated by the exchange of quanta associated with the force field, but these quanta can be considered ~~themselves~~ as independent particles, whose interactions may themselves be mediated by the matter field, so that the classical distinction between matter and force is in a sense eliminated.

In Chapter 13 we shall discuss briefly a totally different alternative to the ~~discussion~~ treatment of elementary particles afforded by quantum field theory. This is the so-called S-Matrix theory, which was introduced originally in 1943, by Heisenberg, and exhibited as its object of study ~~a~~ the so-called scattering matrix, or S-matrix, which connected via its transition amplitudes the initial and final states in the scattering process. The S-Matrix theory did not attempt to analyse what happened during the actual time of interaction, but was concerned in an essentially phenomenological style with the observable features of scattering processes. In fact, the S-Matrix theory was also able to accomodate description of bound states ~~of~~ and resonances by investigation of the singularities in the S-matrix, considered as a function of complex arguments. In the 1960's, an attempt was made to put S-Matrix theory onto an axiomatic basis by specifying certain general properties which the S-matrix must satisfy including properties of analyticity, crossing symmetry and unitarity- these technical terms will be explained in due course (Chapter 13). The S-Matrix theory led to the idea of the so-called 'bootstrap', in which a single self-consistent scheme of particle interactions was envisaged as uniquely determined by the ~~reactions~~ axioms defining the S-matrix. S-Matrix theory was distinguished from quantum field theory ~~by its~~ as embracing a democratic approach to the elementary particles: no particle was to be regarded as more

fundamental than any other particle, and no attempt was made to analyse complex particles as composites made up of constituent particles, but everything was in a state of dynamic interaction. The S-Matrix theory achieved some initial success, but was found to involve totally intractable mathematical problems, and by the early 1970's was largely abandoned in favor of investigating the properties of the newly discovered gauge theories. However, the philosophical problems posed by the bootstrap approach remained of interest in the conceptual foundations of the subject, and we make no apology for including a discussion of these matters in a book devoted to quantum field theory.

Then there is the question of quantum cosmology, the ultimate question about the origin of the universe, and its possible connection with vacuum fluctuations of the relevant quantum fields. This will be considered in Chapter 14.

Fimally, in a brief concluding chapter, we attempt to summarise what can reasonably be learned from quantum field theory about the ultimate nature of matter, its interactions and related metaphysical questions concerning the ultimate furniture of the universe.

Chapter 2

The N-particle Schrödinger Equation

We begin by developing the quantum mechanics of a one-particle system. The states for a quantum mechanical system are represented by unit vectors in a vector space (technically known as a Hilbert space) defined over the field of complex numbers and equipped with an inner product which we denote by $\langle\alpha|\beta\rangle$ for two vectors $|\alpha\rangle$ and $|\beta\rangle$. The norm of ~~the~~ a vector is $(\langle\alpha|\alpha\rangle)^{1/2}$, and is a positive-definite quantity. ~~The states of quantum mechanical systems are identified with unit vectors in this vector space (technically known as Hilbert space) and~~ The states are thus specified up to an arbitrary phase factor, $e^{i\chi}$.

Physical magnitudes that can actually be measured, called observables, are represented by self-adjoint operators ~~in~~ on this vector space. The possible results of measurement are identified with the real eigenvalues of the operator associated with the observable Q. We represent the eigenvector associated with the eigenvalue q_i by the vector $|q_i\rangle$. If $|\psi\rangle$ is the state vector associated with the system, then the probability of the result q_i of measuring Q in the state $|\psi\rangle$ is given by ~~$|\langle q_i|\psi\rangle|^2$~~, where we assume the eigenvalue to be nondegenerate, i.e. associated with a unique eigenvector.

We shall be concerned with developing the Schrödinger equation with ~~the~~ in so-called wave mechanical version of quantum mechanics, in which we replace the abstract state vectors and operators by their representation in terms of

the basis defined by the eigenvalues and eigenvectors of the position operator X. In fact, X does not strictly have any eigenvectors, due to the fact that any possible real value ~~of X~~ is a possible result of measurement, i.e. the spectrum of possible measurement results is a continuous one. This is reflected in the fact that the eigenvectors of the position operator, if they did exist, would have an infinite norm. We shall ignore the mathematical niceties of developing a theory of operators with a continuous spectrum, and treat the operator X and the associated eigenvalues $|x\rangle$ as though they were genuine vectors in our Hilbert space, but satisfy the orthonormality relation $\langle x|x'\rangle = \delta(x-x')$. We note in passing that this way of developing the theory of operators ~~in the~~ with a continuous spectrum, due to Dirac, can be made rigorous in terms of the notion of a rigged Hilbert space, in which vectors of infinite norm are adjoined to the vectors of finite norm, which are elements of the original Hilbert space.

In the representation afforded by $\{|x\rangle\}$, the momentum operator for a single particle, is represented by the differential operator ~~$-i\hbar\partial/\partial x$~~ ~~$-i\partial/\partial x$~~ ~~x~~. The time-independent Schrödinger equation is obtained by replacing in the classical Hamiltonian $H(P,Q)$, the variable Q, representing the coordinate, by the number x, while the canonically conjugate momentum variable P is represented as we have seen, by the differential operator $-i\hbar\partial/\partial x$. Then the eigenvalue ~~of the~~ equation for

Whereas the time-independent equation can be used to discover the possible energy eigenvalue of the system, the time-dependent equation can be used for solving scattering problems in which some initial state is modified by the action of the ~~central~~ potential V, to produce a final state which is the superposition of components, the probability of any one of these components being the result of measurement, ~~is~~ being given by the square of the modulus of the corresponding coefficient in the final superposition of states.

We now want to explain how this formalism is extended to accommodate the quantum mechanics of an N-particle system, in three-dimensional space, in which the N particles are subject to a central potential $V(\underset{\sim}{r})$ and also to a two-particle interaction potential $U(\underset{\sim}{r}_i, \underset{\sim}{r}_j)$ representing the interaction between the ith and the jth particle. Clearly the formalism we shall develop can be extended to multi-particle interactions, but all the features we shall have need to stress ~~discuss~~ arise in the more usual case where the only interaction between the particles is of a two-body character.

To begin ~~Again~~ with the abstract vector space approach to quantum mechanics, in this case the appropriate construction will describe all the states of the N-particle system in the N-fold tensor product of the vector spaces, i.e. Hilbert spaces, appropriate to the individual particles. For example, if $|a^s_n\rangle$ denotes the eigenvector with eigenvalue ~~A~~ a^s for the maximal observable A_n for the n^{th} ~~alpha~~ particle, ~~let us~~ then the

state of ~~change~~ the system for which the ~~alpha~~ n^{th} particle is in the state $|a_n^{s_n}\rangle$ will be given by the N-~~fi~~ld tensor product $|a_1^{s_1}\rangle \cdot |a_2^{s_2}\rangle \cdot \ldots |a_N^{s_N}\rangle$. On this N-~~fi~~ld tensor product space the operator A_n will be extended to the operator $I \otimes I \otimes \ldots \otimes A_n \otimes \ldots \otimes I$, where ~~this~~ the sequence of factors in the N-fold product is in the order of the particle labels 1...N. ~~Assuming for the sake of simplicity that each each particle is described in an n-dimensional Hilbert space, then~~, the general state of the N-particle system is some linear combination of states, such as that just exhibited, for all possible sequences $\{s_n\}$. The expansion coefficient of the state exhibited above in such a general superposition has the interpretation that its modulus squared gives the probability, ~~but~~ that on measurement of A_n ~~a_I~~ for the ~~alpha~~ n^{th} particle, $n = 1 \ldots N$, ~~is the probability that for the alpha particle~~ we shall find the result of measurement to be $a_I^{s_n}$, $a_n^{s_n}$.

We shall now specialise to the case where we take for the observable A_n ~~a_I~~, the position variable $\underset{\sim}{x}_n$ ~~$\underset{\sim}{x}_I$~~ for the ~~alpha~~ n^{th} particle. Introducing the N-particle wave function

~~$\psi(x_1, x_2 \ldots x_n, t)$~~ $\psi(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \ldots \underset{\sim}{r}_N; t)$

defined by ~~the~~ inner product

~~$\langle x_1| \langle x_2| \ldots \langle x_N| \psi \rangle$ and $\psi \langle x_1, x_2 \ldots x_N, t \rangle$~~

$\langle \underset{\sim}{r}_1| \langle \underset{\sim}{r}_2| \ldots \langle \underset{\sim}{r}_N| \psi(t) \rangle$, this

* Question

$$\left[\sum_{i=1}^{N}\left[\left(-\frac{\hbar^2}{2m_i}\nabla_i^2\right)+V(\underset{\sim}{r}_i)\right]+\sum_{j=i+1}^{N}\sum_{i=1}^{N}U(\underset{\sim}{r}_i,\underset{\sim}{r}_j)\right]\Psi(\underset{\sim}{r}_1,\underset{\sim}{r}_2\ldots\underset{\sim}{r}_N;t)$$

$$= i\hbar\,\partial\Psi(\underset{\sim}{r}_1,\underset{\sim}{r}_2\ldots\underset{\sim}{r}_N;t)/\partial t$$

where ∇_i^2 is the usual Laplacian operator for the i^{th} particle

$$\nabla_i^2=\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial y_i^2}+\frac{\partial^2}{\partial z_i^2}$$

Note: p. 31–33 change $\underset{\sim}{r}$ to $\underset{\sim}{x}$, $d^3\underset{\sim}{r}$ to $d^3\underset{\sim}{x}$ and write

$$\nabla_i^2=\frac{\partial^2}{\partial x_{i1}^2}+\frac{\partial^2}{\partial x_{i2}^2}+\frac{\partial^2}{\partial x_{i3}^2}$$

satisfies the time-dependent N-particle Schrödinger equation

$$ih/d*dt\ *\langle x_1,x_2\ldots x_N,t\rangle = [* I=1\ldots N] - h/^2/2N_i*_i{}^2]$$

$$+ v(x_i) + *1$$

$$= 1\ldots N*j$$

$$= i + 1\ldots N \quad \ldots \text{gap in tape} \ldots$$

$$\ldots x_1,x_j]]**(x_1,x_2\ldots x_N,t)$$

where now, $|*|^2\, dx_1,dx_2\ldots dx_N$ $|\psi|^2 d^3\vec{r}_1\, d^3\vec{r}_2 \ldots d^3\vec{r}_N$ now

measures the probability that the measurement will reveal the particle one to have position in the volume element $d^3\vec{r}_1$ around $\vec{r}_1$, $x_1>x_1 + dx_1$, the particle two to have position in the volume element $d^3\vec{r}_2$ around $\vec{r}_2$ interval $x_2>x_2 + dx_2$, etc., etc. It is important to notice this point, that we are assuming that the particles can be experimented on individually, i.e. they can be identified experimentally one from another, so that it makes sense to talk of measuring properties simultaneously on the different particles. We shall discuss in the next chapter the additional complications that arise when the particles cannot be experimentally individuated in the way that we have just assumed. This will raise important questions about problems of identity and individuality for atomic and subatomic systems which will be related to some of the classic philosophical discussions of these topics in the literature.

Chapter 3

Quantum Statistics, Indistinguishable Particles and the Identity of Indiscernables

Quantum statistical mechanics is concerned with the number of states available to an assembly of quantum mechanical systems. This number, W, is then related to the entropy of the system by the usual relation S=klnW. Now Bose in his derivation of the Planck distribution law had already introduced a novel statistical treatment for the light particles, i.e. photons. Effectively, he assumed that the state of an assembly of photons was characterised by the number of particles which occupied each one-particle state, and was in no way concerned with which particles occupied which state, i.e. the state of the assembly obtained by permuting the labels of the particles simply reproduced the ~~smae~~ same state. In the meantime, Fermi in 1925 had introduced another type of quantum statistics, 'Fermi statistics', appropriate to discussing the properties of assemblies of particles such as electrons which obeyed the Pauli Exclusion Principle. This assumed that each quantum mechanical state, specified by ~~an~~ the appropriate set of quantum numbers, could only be occupied by a single particle. In 1926 Heisenberg, and independently ~~of~~ Dirac, showed that these new types of statistics could be accomodated in the quantum mechanical formalism by assuming that the states of assemblies of particles were either symmetric under the

exchange of particle labels, giving rise to Bose statistics, or anti-symmetric under permutation of particle labels, giving rise to Fermi statistics.

We now show formally how to introduce this idea of symmetrising and anti-symmetrising state vectors into the tensor product formalism for discussing the quantum mechanics of N-particle systems. Consider the N-particle state vector,

$$|X\rangle = |a_1^{s(1)}\rangle\ |a_2^{s(2)}\rangle \cdots |a_n^{s(n)}\rangle \cdots |a_N^{s(N)}\rangle,$$

where $s : \{1 \cdots N\} \to I$

is a function from the particle labels to an indexed set I, specifying the eigenvalues ~~A~~ , a^i, $i \in I$ of some single-particle observable A. We now introduce two sorts of permutation operators. First of all, permutation operators P, that permutes the particle labels, and permutations $\bar{P}$ that permute the labels in the indexed set I, ~~i~~, i.e. the state labels. We have that

$$P_i |X\rangle = |a_{i_1}^{s(1)}\rangle \cdots |a_{i_n}^{s(n)}\rangle \cdots |a_{i_N}^{s(N)}\rangle$$

where P_i is ~~an~~ the permutation $\begin{pmatrix} 1 \cdots n \cdots N \\ i_1 \cdots i_n \cdots i_N \end{pmatrix}$

while

$$\bar{P}_i |X\rangle = |a_1^{s(i_1)}\rangle \cdots |a_N^{s(i_N)}\rangle \cdots |a_N^{s(i_N)}\rangle$$

It is clear that P_i commutes with $\bar{P}_j$, for all i and j, and that acting on the ket $|X\rangle$, $\bar{P}_i$ is equivalent to P_i^{-1}. This follows from the fact that $P_i \cdot \bar{P}_i |X\rangle = |a_{i_1}^{s(i_1)}\rangle \cdots |a_{i_N}^{s(i_N)}\rangle \cdots |a_{i_N}^{s(i_N)}$, which of course is exactly the same as the ket $|X\rangle$. It is important to notice, however, that $\bar{P}_i$ can only be identified with P_i^{-1} when acting on the ket $|X\rangle$, which is not in general true that, for example, $\bar{P}_j P_i$ acting on $|X\rangle$ is equal to $P_j^{-1} \cdot P_i$ acting on $|X\rangle$. This follows immediately from the fact that the symmetric group comprising the collection of all the P_i's and their products, is not an Abelian group, so that P_j and P_j do not in general commute.

We now consider how to form the symmetric state,

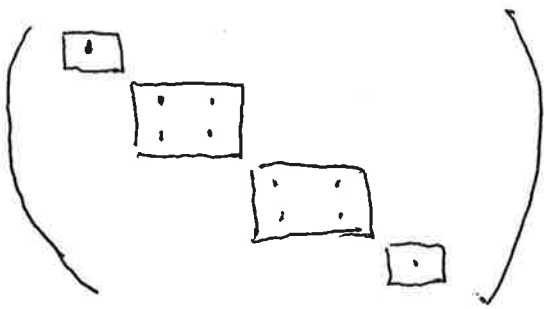
$$S|X\rangle = \sum_i P_i |X\rangle \cdot \frac{1}{\sqrt{n_1! n_2! \cdots} \cdot \sqrt{N!}}$$

Here n_1 is the number of particles in the first distinct state whose label occurs in the range of the function s, and n_2 is the number of particles in the second distinct state whose label occurs in the range of the function s, and so on. The summation is over all the N! permutations that belong to the symmetric group on N labels. The anti-symmetric

combination $A|X\rangle$ is given by the same formula but with a coefficient representing the signature of the permutation, i.e. equals +1 if the permutation is even, -1 ifthe permutation is odd, in front of each term of the summation, and in this case each of the numbers n_1, n_2, etc. must be equal to 1, for if there is more than one particle in the same state, the anti-symmetric combination comes out to be identically zero, i.e. there is no such state available to an assembly of fermions.

If we can set up a collection of states $\{P_i|X\rangle\}$,

then these states clearly support a representation of the symmetric group of permutation operators $\{P_i\}$. However this representation is in general reducible. The two states we have identified are the symmetric and anti-symmetric combinations representing one-dimensional irreducible subspaces for the N-particle system. In the case of two particles, these are the only irreducible representations, but for more than two particles there are additional higher-dimensional representations. For example, in the case of three particles, the representation, which is indeed the regular representation of the symmetric group, reduces to the block diagonal form shown below :



The two-dimensional irreducible representation occurs twice in the reduction of the regular representation. These higher-dimensional representations of the symmetric group played an important role in discussing the properties of so-called 'paraparticles'.

In the philosophical literature, the notions of identity, individuality and indistinguishability have received a great deal of attention. We shall now discuss the relevance of these traditional philosophical arguments in the way these notions are treated in quantum physics.

Identity is a relation that may exist between items that figure in a particular area of discourse. Item a is identical to item b, symbolically a=b, means that infomally there is not in reality two distinct items at all, but only one item, which we may refer to indifferently as a or b. We use 'item' here as a neutral word that comprehends both universals and particulars. Particulars that exist in the physical world we shall refer to as physical individuals, or just individuals for short. The existence of such individuals we take to imply their persistence through time. They are continuants. Physical individuals are distinguished from other sorts of particulars such as events or states of affairs which may be said to occur, or obtain, rather than exist. In the philosophical lierature individuals which have more

or less well-defined spatial locations are often referred to as 'things'.

What confers particularity, or individuality as we shall call it, on physical individuals? This raises the fundamental problem of how the particular is related to the universal, of how ~~the~~ an individual is related to its attributes. Is it possible for two (non-identical) individuals to have all the same attributes in common, that is to say, to be indistinguishable (indiscernible in traditional phiosophical terminology)? Leibniz famously claimed that this was not possible in his Principle of the Identity of Indiscernibles (P.I.I.). In terms of second-order logic with equality, P.I.I. states: ~~for all~~

* $\forall f(f(a) \leftrightarrow f(b)) \rightarrow a = b$ where a and b are any two individual constants and f is a predicate variable, ranging over the possible attributes of these individuals.

* should be contrasted with the P.I.I., that ~~a=b for all f(f(a) f(b)) **~~

** $a = b \rightarrow \forall f(f(a) \leftrightarrow f(b))$

Now ** is uncontroversial, provided the attributes do not involve intentional contexts. If two individuals are identical, so there is in reality just one individual, then there can only be one set of truly predicated attributes. But * has led to a great deal of ~~original~~ argument. What sort of attributes should be included in the range of the variable f? Should we include relations (non-monadic properties)? If we include

being identical with a

the attribute "b=a", which is certainly true of a, then * is a theorem of second-order logic. But suppose we rule out trivialisations of P.I.I. of this sort, on the grounds that such attributes aren't a genuine monadic property, but would express as a relation of a to itself, which relation (of identity) is also, says Herbert, satisfied by b, then we can still distinguish a weak and strong version of P.I.I.

Weak Version: f includes properties of spatial location.

Strong Version: fx excludes properties of spatial location.

Leibniz himself apparently subscribed to the strong version of P.I.I. Hence his interest in searching for indistinguishable leaves in the Herrenhausen grounds in Hanover. If we subscribe to P.I.I. in the weak version, this raises important questions concerning what we understand by about spatial location. On relational theory of space a circularity threatens. Individuation of material objects- leaves, tables, chairs - involves specifying their location in space, but these locations involve their relations with other physical individuals comprising a reference frame. Unless the world is sufficiently asymmetrical for these objects to always bear a set of qualitative spatial relations to all other objects which it shares with no other object, the individuation of the reference frame cannot itself rely on spatial relations. Absolute theories of space avoid this possible circularity, but at the expense of requiring require an account of what it is that confers it individuality on the points of

space. These matters will be given further discussion in Chapter 5 below.

P.I.I. seeks to reduce the particular to the universal, the individual to a bundle of properties or attributes. But many philosophers would argue that such a reduction is not possible, that the individual is involved in something over and above the attributes, that confers individuation in an essential and unanalysable way. This is to go the way of the Lockean substratum, the unknowable "something" that attributes "attach" to.

If the individual acquires its individuality by something that transcends its attributes, we shall say it exhibits 'transcendental individuality', T.I. for short. It is fair to say that modern analytical philosophy has not taking kindly to T.I. Physical individuals are usually regarded as being individuated by their location in space, despite the problem we've just referred to (which may threaten to re-introduce T.I. for points of space!), and the problem of re-identification, i.e. the grounds of claiming that individual b at time t is the same individual as the individual a at an earlier time t', involves the necessary condition of spatio-temporal continuity of the trajectory, joining the location of a at time t', and the location of b at time t. The proponents of T.I. might agree that spatio-temporal continuity (S.T.C.) is what allows us to infer a re-identification across an interval of time, but they would

claim that it is the persistent T.I., that, ontologically speaking, confers the re-identifiability.

In classical 19th century atomic physics the strongest version of P.I.I., with attributes restricted to monadic, intrinsic properties, i.e. 'internal' properties independent of spatial location, is clearly ~~true of~~ false for the atoms. The weak version is ~~also~~ however true, ~~in view of~~ if we rely on the Impenetrability Assumption (I.A.), ~~to the effect~~ that two distinct atoms can ~~not~~ never occupy the same location in space. We want to begin our discussion of quantum physics ~~with~~ by considering an argument to the effect that quantum particles cannot be regarded as individuals at all. If this were the case, the problem of how they were individuated simply would not arise. P.I.I. would not be either true or false, but simply inapplicable. The argument runs like this. Consider the problem of distributing two quantum particles having the same intrinsic properties such as mass, spin and charge, but initially supposed to be individuals and labelled one and two, among two possible pure quantum states $|a^r\rangle$ and $|a^s\rangle$ which we may suppose to be eigenstates of a maximal observable ~~a~~ A of either particle, with eigenvalues a^r and a^s indicated by the notation ~~of~~ for the states. We assume for simplicity that all observables under discussion have a discrete spectrum. By analogy with the situation in classical physics we ~~may~~ might suppose that there are four possibilities:

1) both particles are in the state $|a^2\rangle$

2) both particles are in the state $|a^5\rangle$

3) particle 1 is in the state $|a^2\rangle$
and particle 2 is in the state $|a^5\rangle$

4) particle 1 is in the state $|a^5\rangle$
and particle 2 is in the state $|a^n\rangle$

Now, In classical statistical mechanics, arrangements 3 and 4 would be counted as distinct, and given equal weight in assigning probabilities. In quantum statistics, whether bosonic or fermionic, the arrangements 3 and 4 counts as one and the same arrangement for the purposes of assigning weights. This is taken to show that the two arrangements ~~underlined~~ are not only indistinguishable but actually identical, but ontologically speaking these two arrangements are not identical if the two quantum particles are individuals, hence the quantum particles cannot be individuals.

In passing we may note that this argument, while purporting to show that quantum particles fall outside the scope of P.I.I., since they're not individuals, is also sometimes invoked to show that P.I.I. does apply to the states of affairs represented by the two arrangements 3 and 4.

What about ~~the~~ arrangements 1 and 2? This is where bosons differ from fermions. For ~~The~~ bosonic particles 1 and 2 are allowed arrangements to be counted with equal weight, as compared with the single identified three-cum-four

arrangement. For fermionic particles, however, arrangements 1 and 4 are not permitted at all, as a result of the Pauli Exclusion Principle.

We shall return shortly to discuss the significance of this difference, from the point of view of P.I.I., but first we want to explain what is wrong with the argument concerning the identity of arrangements 3 and 4. We begin by writing down the state vector for the combined two-particle system corresponding to the arrangements 1, 2, 3 and 4. They are:

(1) $|a^r\rangle \cdot |a^s\rangle$

(2) $|a^s\rangle \cdot |a^s\rangle$

(3) $|a^r\rangle \cdot |a^s\rangle$

(4) $|a^s\rangle \cdot |a^r\rangle$

where we use the convention that in the tensor product of two states, the left-hand member refers to particle 1 and the right refers to particle 2.

Now it is quite true that if the quantum particles are individuals, then the states three (3) and four (4) are not identical. But an important point to notice is that these states are not the one used for discussing quantum staistical mechanics. The relevant states for that purpose are as follows:

(5) $|a^n\rangle \cdot |a^{n2}\rangle$

(6) $|a^s\rangle \cdot |a^s\rangle$

(7) $\frac{1}{\sqrt{2}}\left(|a^{n2}\rangle \cdot |a^s\rangle + |a^s\rangle \cdot |a^n\rangle\right)$

(8) $\frac{1}{\sqrt{2}}\left(a^{|n\rangle} \cdot |a^s\rangle - |a^s\rangle \cdot |a^{n2}\rangle\right)$

The four states, (5), (6), (7) and (8), are mutually ~~diagonal~~ orthogonal and span the same subspace as the states (1), (2), (3) and (4). But they're chosen so that (5), (6), and (7) are symmetric under exchange of particle labels, i.e. under exchange of left-hand and right-hand members of tensor products, while (8) is anti-symmetric (changes sign) under the same operation.

Note that (5) and (6) are the same states as (1) and (2). The crucial difference between the pairs (3) and (4), and (7) and (8).

Now (7) is no more identical with (8) than is (3) with (4). But for bosons the states are restricted to the three symmetric possibilities. That is why (8) gets eliminated from the counting procedure, not because it gets identified with 7. Similarly, for fermions the states are restricted to the anti-symmetric possibilities. But in this simple example, this eliminates (5), (6) and (7), so (8) alone gets counted, but again not because it gets identified with (7).

Putting the matter another way, states of the wrong symmetry get eliminated because they're not accessible to the quantum system, not because there are no such states. To make this point clear, we shall show in a moment that for time evolution under a symmetric Hamiltonian, the symmetry character of a state cannot change with time, so no transition can occur between symmetric bosonic states and anti-symmetric fermionic states. In other words, if the initial state of a bosonic system is symmetric, it remains permanently symmetric, while if the initial state of a fermionic system is a-symmetric, it remains permanently anti-symmetric.

The upshot of this argument is to show, not that quantum particles must be individuals, but rather it is possible for them to be individuals despite the peculiarities of quantum statistics.

What do we mean by saying that two quantum particles of the same species (characterised by their intrinsic properties) are indistinguishable?

In quantum mechanics (Q.M.), this is expressed by the Indistinguishability Postulate (IP)

(9) $\langle P\phi|Q|P\phi\rangle = \langle\phi|Q|\phi\rangle$

for all Q, and for all $|\phi\rangle$

where $|\phi\rangle$ is an arbitrary N-particle state, and Q a possible observable on the N-fold tensor product of spatial states. $|P\phi\rangle$ is an abbreviation for $P|\phi\rangle$ where P is the operator which is associated with an arbitrary permutation of the particle labels. (9) says that it is not possible to tell by measuring the expectation value of any observable, whether the state of a system is $|\phi\rangle$ or $|P\phi\rangle$.

We notice that as a sufficient condition for (9) to hold, is that $|P\phi\rangle = \pm |\phi\rangle$ with Q any operator on the N-particle state space. This interprets (9) as a restriction on the possible states for the N-particle system, allowing just the boson or fermion possibility (note that the choice of signs needs only to be made for transpositions, since any permutation can be represented as a product of transpositions, so even permutations are always associates with the + sign, the distinction between bosons and fermions only arising from odd permutations).

But Greenberg and Messiah in 1964 pointed out that (9) should, on a more profound analysis, be interpreted not as a restriction on states, but as a restriction on the possible observables for the N-particle system. On this view (9) can easily be shown to imply

$$P^{-1} Q P = Q$$

or $QP = PQ$,

So any permitted Q must commute with any permutation P. This in turn implies that Q must be a symmetric function of the particle labels. The label permutations provide, effectively, a set of non-Abelian superselecting operators, which can be used to reduce the state space to the direct sum of the non-combining sectors associated with irreducible representations of the symmetric group S_N. As we have already indicated, for two particle there are only two irreducible representations of S_2, provided by states which are symmetric or antisymmetric under permutation of particle labels, so we are back to just the boson and fermion possibilities. But with more than two particles higher-dimensional representations of the symmetric group exist, allowing for the possibility of so-called parastatistics, intermediate in character between bosonic and fermionic behaviour.

We should notice in particular that since the P's all commute with the $\overline{P}$'s, it follows that the $\overline{P}$'s are themselves permitted observables, although the P's are not, since a collection of all the P's is as we've seen non-Abelian, i.e. any particular P will not in general commute with every other P. Since the $\overline{P}$'s are concerned with permuting states rather than particles, it is perfectly reasonable that the substitution of one state for another may make an observable difference, whereas substitution of one particle for another does not. The conceptual distinction between the particle label permutation

operators P and the state label permutation operators $\bar{P}$ is very important for the whole discussion.

It is very important to notice that even in the two-particle case, the Messiah-Greenberg approach does not restrict the available states, only their accessibility, in the way we described above, due to the absence of possible transitions between states supporting different irreducible representations.

Let Q now denote a possible observable (self-adjoint operator) on a single particle. Considering the possible physical magnitude for the joint system, we have two possibilities, $Q \otimes I$ for particle 1 having the property Q, and $I \otimes Q$ for particle 2 having the property Q. The Indistinguishability Postulate says, on the Messiah-Greenberg interpretation, that although Q_1 and Q_2 are self-adjoint operators on the Hilbert space for the the joint system, they cannot actually be observed. The intuition here is that observing Q_1 or Q_2 would involve knowing empirically which particle is which, and this is impossible if the particles are indistinguishable. But from the point of view of discussing P.I.I., it seems clear that we should not restrict the discussion to attributes that can actually be observed, as this would restrict us to symmetric combinations such as $Q_1 + Q_2$. The underlying ontological significance of P.I.I. can only be brought out by discussing whether particles 1 and 2 have the same physical attributes expressed by Q_1 and Q_2

We denote $Q \otimes I$ by Q_1 and $I \otimes Q$ by Q_2

and ~~the~~ their associated 'actualisation' probabilities, while recognising that the attribute can never be observed. We refer to 'actualisation' rather than 'measurement result', to emphasise the point that they are not observable. They will, however, be produced by measurement interactions, but not in a way which makes them identifiable as ~~contrasting~~ contrasted with their particle label permuted variance. But note that the probability of observing some eigenvalue q^{α} for Q on one or other underlying particle is calculated in Q.M. as ~~a~~ the sum, $\mathrm{Prob}(Q_1 = q^{\alpha}) + \mathrm{Prob}(Q_2 = q^{\alpha}) - \mathrm{Prob}(Q_1 = q^{\alpha} \,\&\, Q_2 = q^{\alpha})$. We are here claiming, then, ontological significance for each individual term in this formula for an observable probability.

We begin our discussion with the case of fermions. It has been claimed in the literature that the Pauli Exclusion Principle (P.E.P.), ~~prohibited~~ prohibiting two fermionic particles from being in the same quantum state, ~~in~~ is a clear violation of P.I.I. What ~~it's indeed~~ being prohibited, apparently, is that the two fermions ~~will~~ shall have both the same intrinsic state-independent properties of mass, spin, electric charge, etc., and the same state-dependent properties expressed by expectation values of all quantum-mechanical physical magnitudes. But in the allowed state (8), it is not true to say that each particle is present in a different state. Each particle clearly 'partakes' of both underlying states $|a^r\rangle$ and $|a^s\rangle$ in the superposition of states expressed in (8). So might it not appear that in the allowed state, both particles also

have the same underlying state-dependent properties, which would contradict P.I.I.? Let us formulate the state-dependent properties in terms of physical magnitudes such as Q_1 and Q_2 pertaining to each particle separately, as discussed in the preceding section, then in orthodox interpretations of quantum mechanics the properties Q_1 and Q_2 must be interpreted not as possessed values, but as propensities to yield specified 'actualisation' results in accordance with the familiar statistical algorithm for computing the associated probabilities. Denoting the fermion state by $|\Psi\rangle$ we shall be interesting/concerning monadic properties of the form

$$Prob^{|\Psi\rangle}(Q_1 = q^\alpha)$$

and

$$Prob^{|\Psi\rangle}(Q_2 = q^\alpha),$$

where the notation indicates the probability that in the state $|\Psi\rangle$ (close up) the physical magnitude pertaining to either particle and actualises with the indicated value, also relational properties of the form

$$Prob^{|\Psi\rangle}(Q_1 = q^\alpha / Q_2 = q^\beta)$$

and

$$Prob^{|\Psi\rangle}(Q_2 = q^\alpha / Q_1 = q^\beta),$$

which refers to the conditional probability of actualising one magnitude, given the actualisation ~~was~~ result of ~~after~~ the other.

These quantities are easily computed from the joint distribution

$$\text{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha \,\&\, Q_2 = q^\beta) = |\langle q^\alpha|\langle q^\beta|(|\Psi\rangle)|^2$$

$$= \tfrac{1}{2}|\langle q^\alpha|a^n\rangle|^2 \cdot |\langle q^\beta|a^s\rangle|^2 + \tfrac{1}{2}|\langle q^\alpha|a^s\rangle|^2 \cdot |\langle q^\beta|a^n\rangle|^2 - \text{Re}\langle a^n|q^\alpha\rangle\langle q^\alpha|a^s\rangle\langle a^s|q^\beta\rangle\langle q^\beta|a^n\rangle \tag{10}$$

Summing ~~the conditional~~ this result over α and β to obtain the marginal probabilities, and remembering $\sum_n |q^n\rangle\langle q^n| = I$,

$\langle a^r|a^s\rangle = 0$ and $\langle a^n|a^n\rangle = \langle a^s|a^s\rangle = 1$

yields immediately

$$\text{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha) = \text{Prob}^{|\Psi\rangle}(Q_2 = q^\alpha) = \tfrac{1}{2}|\langle q^\alpha|a^n\rangle|^2 + \tfrac{1}{2}|\langle q^\beta|a^s\rangle|^2 \tag{11}$$

Similarly we find

$$\text{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha / Q_2 = q^\beta) = \text{Prob}^{|\Psi\rangle}(Q_2 = q^\alpha / Q_1 = q^\beta)$$

$$= [|\langle q^\alpha | a^\gamma \rangle|^2 \cdot |\langle q^\beta | a^\delta \rangle|^2 + |\langle q^\alpha | a^\delta \rangle|^2 \cdot |\langle q^\beta | a^\gamma \rangle|^2$$
$$- 2\,\mathrm{Re}\langle a^\gamma | q^\alpha \rangle \langle q^\alpha | a^\delta \rangle \langle a^\delta | q^\beta \rangle \langle q^\beta | a^\gamma \rangle]$$
$$/ [\,|\langle q^\beta | a^\gamma \rangle|^2 + |\langle q^\beta | a^\delta \rangle|^2\,] \qquad (12)$$

The significance of (11) and (12) is that the two fermions in the state (8) do in fact have the same monadic properties and the same relational properties one to another, so the weakest form of P.I.I. which we can formulate, which involves both monadic and ~~underlying~~ relational properties, is violated. This result remains true if we consider also relational properties of the form $Prob^{|\Psi\rangle}(Q_1 = q^\alpha / Q_2' = q'^\beta)$

which is easily seen to be equal to
$Prob^{|\Psi\rangle}(Q_2 = q^\alpha / Q_1' = q'^\beta)$

where $Q_1' = Q' \otimes I$

and $Q_2' = I \otimes Q'$

and Q' is a.. self-adjoint ...operator distinct from Q. This generalisation also applies to the other violations of P.I.I. which we are about to discuss.

There are a number of comments we want to make concerning this conclusion and the way it was derived.(1) ~~Now~~ In classical physics the state-dependent properties of a particle are completely specified by the maximum all ~~in a specific~~ state description, i.e. the location in ~~face~~ phase

space. Hence we can replace the question, "do classical particles have the same state-dependent properties?", with the question, "do the two particles have the same maximally specific state descriptions ?". If we try the same move in quantum mechanics, we run into the problem that for so-called 'entangled' states, such as (8), there are no pure states which can be ascribed to the separate particles. This follows since if there were such states, the state for the combined system would be the tensor product of the states in question, but (8) is not the former tensor product - it is a superposition of tensor products.

Now pure states in Q.M. play the role of maximally specific states, so if we identify the relevant properties of the two particles with the pure states they are in, we would have to conclude that there is no answer to the question, "do they have the same properties?".

A corollary of this result is that insofar as we can speak of states for the separate particles at all, we must speak of mixed states. According to the terminology of D'Espagnat, these are so-called 'improper' mixtures, which reproduce correctly the statistics of observables applying to the individual particles, but fail to reproduce correlation properties between the two particles. It is easily shown that the relevant

(improper) mixed states are the same for the two particles - equiprobable mixtures of the states $|a^r\rangle$ and $|a^s\rangle$.

~~and~~

~~This~~ this is, of course, ~~an~~ the essential content of the result ~~of~~ for the marginal probability distributions for Q_1 and Q_2. But our analysis has gone beyond that~~,~~ involving the improper mixed states of the separate particles, by considering also the relational conditional probabilities given in (12). It should be stressed that these relational properties expressed by the conditional probabilities in no way supervene on the monadic properties expressed by the marginal distributions. In the terminology of Teller [1986], they are inherent relations.

(close up)

~~(25~~

(2) There is another sort of relational property one might consider, expressed by comparing

(13) $\text{Prob}^{|\Psi\rangle}(Q_2 = q^\alpha \,/\, Q_2 = q^\beta) = \delta_{\alpha\beta}$

with $\text{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha / Q_2 = q^\beta)$ given by (12).

These relational properties of particle 2 ~~for~~ to itself compared with the relation of particle 1 to particle 2 we reject as vindication of P.I.I., for the same argument as we discussed above for ruling out illegitimate trivialisations of P.I.I. The purported vindication of P.I.I. again depends on regarding (13) as a monadic property of particle 2, whereas it is in fact a relational property of particle 2 to itself, which is also true as a relation of particle 1 to itself.

(close up ~~(13)~~)

(3) ~~If~~ we write $\alpha = \beta$ in equation (12) then we indeed find for fermions the result

$$\text{Prob}(B_1 = q^\alpha \,/\, Q_2 = q^\alpha) = 0$$

This shows that if actualistion of Q_2 gives ~~us~~ a certain value then there is zero probability that a concurrent actualisation of Q_1 will yield the same value. This is the real ~~content~~ significance of the Exclusion Principle, but has no bearing on P.I.I., if we adhere to the ~~un~~orthodox view that actualisations do not correspond to existing possessed values.

(close up ~~(14)~~)

(4) This brings us to our final comment. In hidden-variable reconstructions of Q.M., the circumstance demonstrated in point (3) above, would lead us to regard P.I.I. as

vindicated for fermions, if we assume that actualisation results merely reveal pre-existing values.

We now turn to the case of bosons. As often assumed, the purported violations of P.I.I. depend on consideration of states such as (5) or (6), where both particles can indeed be attributed the same pure state.

Denoting the state (5) by $|\Phi\rangle$, for example, we can easily obtain the following results corresponding to (11) and (12).

(11') $Prob^{|\Phi\rangle}(Q_1 = q^\alpha) = Prob^{|\Phi\rangle}(Q_2 = q^\alpha) = |\langle q^\alpha | a^R \rangle|^2$

and

(12') $Prob^{|\Phi\rangle}(Q_1 = q^\alpha / Q_2 = q^\beta) = Prob^{|\Phi\rangle}(Q_2 = q^\alpha / Q_1 = q^\beta) = |\langle q^\alpha | a^R \rangle|^2$

So, as one might expect, both monadic and relational properties are the same for the two particles. But it should be noted that this conclusion is also true for state (7), where two different states are involved. In this case, the results of (11) and (12) apply with the minus sign in front of the 'interference' term in (12) replaced by a plus sign.

Finally we make a brief comment on the case of paraparticles. Here there do exist states for which the monadic properties of all the separate particles are not

the same, but equally there are possible paraparticle states, for which P.I.I. is violated in the same way as for bosons and fermions. As an example, consider the following normalised state for three ~~parabosons~~ ~~of order two~~ where parafermions

$$(14)\quad |\Psi'\rangle = \frac{1}{\sqrt{2}}\left(|a^r\rangle|a^r\rangle|a^s\rangle - |a^s\rangle|a^r\rangle|a^r\rangle\right)$$

where $|a^r\rangle$ and $|a^s\rangle$ are two distinct one-particle states, and triple tensor products are written in the sequence of particle labels 1, 2 and 3.

Denoting $Q\otimes I\otimes I$ by Q_1, $I\otimes Q\otimes I$ by Q_2 and ~~The node~~ $I\otimes I\otimes Q$ by Q_3, ~~and~~ we obtain for the triple-joint distribution

$$\begin{aligned}
&P_{\mathrm{Rob}}^{|\Psi'\rangle}(Q_1=q^\alpha \& Q_2=q^\beta \& Q_3=q^\gamma)\\
&= |\langle q^\alpha|\langle q^\beta|\langle q^\gamma|(|\Psi'\rangle)|^2\\
&= \tfrac{1}{2}\left[|\langle q^\alpha|a^r\rangle|^2\cdot|\langle q^\beta|a^r\rangle|^2\cdot|\langle q^\gamma|a^s\rangle|^2\right.\\
&\quad + |\langle q^\alpha|a^s\rangle|^2\cdot|\langle q^\beta|a^r\rangle|^2\cdot|\langle q^\gamma|a^r\rangle|^2\\
&\quad \left. - 2\,\mathrm{Re}\,\langle a^r|q^\alpha\rangle\langle q^\alpha|a^s\rangle\langle a^r|q^\beta\rangle\langle q^\beta|a^r\rangle\langle a^s|q^\gamma\rangle\langle q^\gamma|a^r\rangle\right]
\end{aligned}$$

(15)

From ~~On~~ (15) we find immediately the marginal distributions

(16) $$\text{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha) = \text{Prob}^{|\Psi\rangle}(Q_3 = q^\alpha) = \frac{1}{2}\left(|\langle q^\alpha|a^r\rangle|^2 + |\langle q^\alpha|a^s\rangle|^2\right)$$

while

(17) $$\text{Prob}^{|\Psi\rangle}(Q_2 = q^\alpha) = |\langle q^\alpha|a^r\rangle|^2.$$

Thus ~~as~~ particles 1 and 3 have the same monadic property ~~y~~ ips expressed ~~on~~ by the marginal distributions, ~~for~~ but these differ from the ~~varied~~ monadic properties of particle 2. [Let us now show that particles 1 and 3 also have the same relational properties with respect to ~~most of~~ both the remaining particles. We easily find that

$$\text{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha / Q_3 = q^\gamma) = \text{Prob}^{|\Psi\rangle}(Q_3 = q^\alpha / Q_2 = q^\gamma)$$

(18) $$= \left[|\langle q^\alpha|a^r\rangle|^2 \cdot |\langle q^\gamma|a^s\rangle|^2 + |\langle q^\alpha|a^s\rangle|^2 \cdot |\langle q^\gamma|a^r\rangle|^2 - 2\,\text{Re}\langle a^r|q^\alpha\rangle\langle q^\alpha|a^s\rangle\langle a^s|q^\gamma\rangle\langle q^\gamma|a^r\rangle\right] / \left[|\langle q^\alpha|a^r\rangle|^2 + |\langle q^\alpha|a^s\rangle|^2\right]$$

Furthermore,

$$\text{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha / Q_2 = q^\beta)$$

(19) $$= \text{Prob}^{|\Psi\rangle}(Q_3 = q^\alpha / Q_2 = q^\beta)$$

$$= \frac{1}{2}\left[|\langle q^\alpha|a^r\rangle|^2 + |\langle q^\alpha|a^s\rangle|^2\right]$$

and finally,

$$\text{Prob}^{|\Psi\rangle}(Q_1 = q^\alpha / Q_2 = q^\beta \,\&\, Q_3 = q^\gamma)$$

$$= \text{Prob}^{|\Psi\rangle}(Q_3 = q^\alpha / Q_2 = q^\beta \,\&\, Q_1 = q^\gamma)$$

$$= \left[|\langle q^\alpha|a^r\rangle|^2 \cdot |\langle q^\beta|a^s\rangle|^2 \cdot |\langle q^\gamma|a^r\rangle|^2 + |\langle q^\alpha|a^s\rangle|^2 \cdot |\langle q^\beta|a^r\rangle|^2 \cdot |\langle q^\gamma|a^r\rangle|^2\right.$$

(20) $-2\,\mathrm{Re}\langle a^2|q^2\rangle\langle q^2|a^5\rangle\langle a^5|q^3\rangle\langle q^3|a^2\rangle\langle a^2|q^\gamma\rangle\langle q^\gamma|a^2\rangle]$
$/\,|\langle q^\beta|a^2\rangle|^2\,[|\langle q^\gamma|a^2\rangle|^2 + |\langle q^\gamma|a^5\rangle|^2]$

These results show that P.I.I. is violated for particles 1 and 3 in the state $|\Psi'\rangle$, even in its weakest form, that includes ~~that it lose~~ all ~~its~~ the relevant relational properties.

There are two main conclusions that we ~~ought~~ want to draw. Firstly, that indistinguishable particles in Q.M. can be treated as individuals, but secondly, if they are so treated, then, ~~are~~ or the most plausible ~~. . . won't~~ reading of what constitutes a property of a ~~quantum~~ quantal particle, even the weakest form of P.I.I., including both monadic and relational properties, is violated both for bosons and fermions, and indeed for higher-order paraparticles.

It should be noted that if ~~quantum~~ quantal particles are individuals, then the individuality must be ~~confirmed~~ erred by T.I.. S.T.C. is not, in general, available in Q.M., since particles do not move in well-defined trajectories, so the question of spatio-temporal continuity of ~~trajectories~~ trajectory does not arise. The only exception to this

is where the one-particle states involve well-defined wave packets, which diffuse sufficiently slowly over time, as will be possible for the classical limit of sufficiently massive particles.

But it is clear that in the case of macroscopic bodies, where S.T.C. can be used to label the bodies, the S.T.C. criterion actually conflicts with the T.I. individuation of the elementary particles composing the body. To be strict, every electron, for example, partakes of the state of every other electron in the universe (~~call it~~ according to the anti/symmetrisation requirement!).

But notice, that under conditions where the 'interference' term in (10) can be neglected, then the state $|\Psi\rangle$ behaves like a proper mixture of states, in which particle 1 is in state $|a^r\rangle$ and particle 2 is in state $|a^s\rangle$, and the permuted state in which particle 1 is in the state $|a^s\rangle$ and particle 2 is in the state $|a^r\rangle$, with equi ~~then equally~~ probable weights for the two component states in the mixture. So under these conditions, the state (8) behaves like an equiprobable mixture of the states (3) and (4). In other words, when 'interference' can be neglected, we recover the same possibilities for states as in classical physics, where states (1) and (2) would anyway be eliminated by I.A., assuming them to be maximally specific.

But, of course, ontologically speaking, 'interference' is never strictly absent. That, after all, is what constitutes the problem of measurement in Q.M., so the involvement of every electron with the state of every other electron in the universe, although negligible for practical purposes, remains ~~in~~ an ontological commitment of Q.M., under the interpretation where the particles are treated as individuals.

If this sounds too bizarre to be acceptable, then we shall find in the discussion of quantum field theory a much more straightforward way of dealing with the problem of indistinguishable particles.

For the moment we've been concerned with conceptual possibilities, rather than what is most reasonable to believe about the ontological status of elementary particles.

Chapter 4

Second Quantization

We begin by considering the case of indistinguishable bosons. We shall show how a mathematical reformulation of the N-particle Schrödinger equation, subject to the symmetrisation condition for the bosonic state vector, leads to the so-called 'second quantized' version of boson theory.

Let us choose for the observable A in the state vector $|X\rangle$, introduced on p. 35, the Hamiltonian operator for a single particle H. The Hamiltonian for the assembly of N non-interacting systems is $\sum_{n=1}^{N} H_n$, and it is clear that $S|X\rangle$ is an eigenvector of this total Hamiltonian, with the eigenvalue $\sum_s n_s E_s$, i.e.,

$$\left[\sum_{n=1}^{N} H_n\right] S|X\rangle = \left(\sum_s n_s E_s\right) S|X\rangle$$

It is also clear that $S|X\rangle$ is specified entirely in terms of the number of particles occupying each one-particle state, i.e. it can be represented in the form

$$|n_1, n_2 \cdots n_s \cdots\rangle$$

We now draw a formal analogy with another quantum-mechanical assemblage, which has very nearly the same eigenvalue spectrum, as we have just exhibited for a system of N non-interacting bosons. This is an assembly of N' harmonic oscillators where N' is the number of distinct one-particle states in the vector $|X\rangle$. For one harmonic oscillator, the energy spectrum is well known to be of the form $\left(n_s + \frac{1}{2}\right)\hbar\omega_s$ where ω_s is the angular frequency of the harmonic oscillator, and for the assembly of N' independent harmonic oscillators the energy spectrum is clearly of the form $\sum_{s=1}^{N'}\left(n_s + \frac{1}{2}\right)\hbar\omega_s$. Now this is exactly the same as the energy spectrum for the system of bosons, except for the term $\sum_{s=1}^{N}\frac{1}{2}\hbar\omega_s$. But since it is only changes in energy which are of significance, we can alter the zero from which we measure our energies by subtracting off this so-called zero-point energy, and can

then proceed to identify the energy spectrum of ~~the~~ an assembly of N bosons with an assembly of N' independent harmonic oscillators. ~~Clearly, the ket vector $S|\chi\rangle$ is completely specified by the number of particles occupying each distinct one-particle state, i.e. can be rewritten as $|n_1, n_2 \cdots n_s \cdots\rangle$~~ Moreover, we can now eliminate the restriction to N' possible states, and consider the complete spectrum of the possible one-particle states, on the understanding that the relevant value of n_s for ~~this~~ a state which is unoccupied, is simply zero. And thus we can extend the specification of our state, in the form $|n_1, n_2 \cdots n_s \cdots\rangle$, where we now include all values of ~~s~~ s which label the possible one-particle states. We assume a discrete spectrum, but the number of states may well be infinite. Notice that in the new specification of states, the particle labels have disappeared. The subscript s now labels the particle states. So it makes sense to talk about permutations of state labels, but permutations of particle labels make no sense with our new notation.

At this point, we can exploit the well-known mathematical machinery for dealing with the quantum mechanics of ~~a~~ the harmonic oscillator to make some significant transformation of our theory for the system of N non-interacting bosons. It is well known that the quantum mechanics for a harmonic oscillator is much simplified by

introducing the so-called raising and lowering operators, a_s^+ and a_s, which have the following action on the state $|n_s\rangle$ with quantum number n_s, namely

$$a_s^+ |n_s\rangle = \sqrt{n_s+1}\ |n_s+1\rangle$$

and

$$a_s |n_s\rangle = \sqrt{n_s}\ |n_s-1\rangle$$

The operators a_s^+ and a_s satisfy the commutation relations $[a_s, a_{s'}^+] = \delta_{ss'}$, and it is easily shown that the operator $N_s \underset{Df}{=} a_s^+ a_s$ has eigenvalues $0, 1, 2$, etc. Thus N_s acting on $|n_1, n_2 \cdots\rangle$ is just $n_s |n_1, n_2 \ldots\rangle$.

$[a_s, a_{s'}] = [a_s^+, a_{s'}^+] = 0$

Now E_s is simply the expectation value of the one-particle Hamiltonian in the state $|E^s\rangle$, i.e. is equal to the integral $\int \psi_s^*(\underset{\sim}{x}) H \psi_s(\underset{\sim}{x})\, d^3\underset{\sim}{x}$ where $\psi_s(\underset{\sim}{x})$ is the wave function associated with the state $|E^s\rangle$. Thus we can write $\sum_{n=1}^{N} H_n$ as equivalent to $\sum_s N_s \int \psi_s^*(\underset{\sim}{x})\, H\, \psi_s(\underset{\sim}{x})\, d^3\underset{\sim}{x}$

$$= \int \Psi^+(\underset{\sim}{x})\, H\, \Psi(\underset{\sim}{x})\, d^3\underset{\sim}{x}$$

where $\Psi(\underset{\sim}{x})$ is equal to $\sum_s a_s \psi_s(\underset{\sim}{x})$ and similarly $\Psi^+(\underset{\sim}{x}) = \sum_s a_s^+ \psi_s^*(\underset{\sim}{x})$.

Since any arbitrary state can be expanded as a suitable linear combination of states such as $\$|X\rangle$, it follows that this is a general representation for the operator of the form $\sum_{n=1}^{N} H_n$, acting on the space of symmetric states for the N-particle system, i.e. on the collection of states spanned by $|n_1, n_2 \ldots\rangle$.

note: pp. 65-66 change $\psi_s(\underset{\sim}{x})$ to $u_s(\underset{\sim}{x})$ and change $\underset{\sim}{x}$ to $\underset{\sim}{n}$ for rest of Ch 4

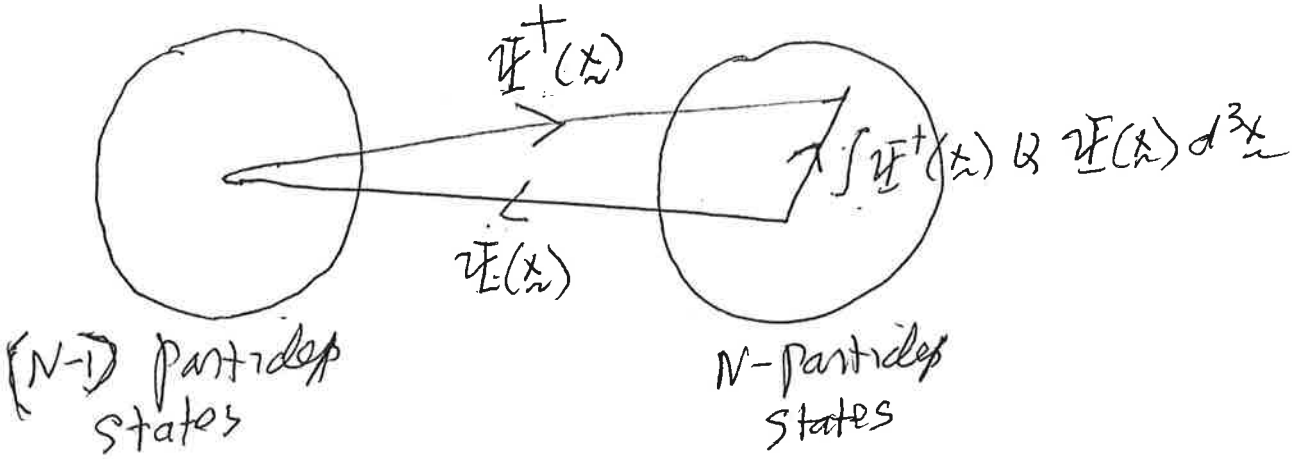
Furthermore we should notice that this representation of $\sum_{n=1}^{N} H_n$ does not depend on the expansion in terms of the eigenstates of energy $\psi_s(x)$ but is equally well true if we expand $\Psi(x)$ in the form $\sum_s b_s \nu_s(x)$ where $\nu_s(x)$ is some arbitrary wave function which is a member of an orthonormal set $\{\nu_s\}$ and in terms of which $\psi_s(x)$ can be expanded in the form $\sum_t C_{st} \nu_t(x)$. It is clear that the commutation properties of the b's are exactly the same as for the a's, and it follows moreover that any operator in the original problem of the form $\sum_{n=1}^{N} Q_n$ can be represented in the form $\int \Psi^+(x) Q \Psi(x) d^3x$, where Q is the one-particle operator. Looking at the above expansion formula for $\Psi(x)$, it is clear that $\Psi(x)$ is formally analogous to a wave function, which can be expanded in terms of any orthonormal set of functions. But it is also clear that this 'wave function' is now an operator, since the expansion coefficients a_s are themselves operators. This representation of operators such as $\sum_{n=1}^{N} Q_n$ is known as the second quantized representation. The term 'second quantization' comes from the fact that the wave function-like objects $\Psi(x)$ are now interpreted as operators. But on what space do these operators act? We have to introduce the notion of what is called Fock space, which is a direct sum of Hilbert spaces for zero-particle, one-particle, two-particle, etc. systems. The reason for this is clear, since operators of the form $A\, a_s$ take us from the N-particle sector of the Fock space down

to the N-1 particle sector of the Fock space, ~~where~~ while operators such as ~~a~~ a_5^+ take us from the N-particle sector of the Fock space to the N+1 particle sector of the Fock space, but.. note that at the whole operator $\int \Psi^+(x) \Psi(x) d^3x$ ~~dx~~ leaves us with a vector in the same sector of Fock space as the one in which we start. Essentially, we are taking a kind of square root of operators acting on the individual sectors of the Fock space. This is illustrated schematically in the diagram below. (see p. 68) $\Psi(x)$ takes us down from the N-particle sector of the Fock space to the N-1 particle sector, and then $\Psi^+(x)$ takes us up again into the N-particle sector. So the net transformation effected by the integral operator is simply to move us from one location ~~of~~ in the N-particle sector of Fock space to another location in the N-particle sector of Fock space.

We mention without proof that the two-particle interaction term in the N-particle Schrödinger equation with interparticle interaction can also be represented by our Fock space formalism, thus

$$\sum_{j=1}^{N} \sum_{i>j}^{N} U(x_i, x_j) = \frac{1}{2} \int\int \Psi^+(x_i) \Psi^+(x_j) U(x_i, x_j) \Psi(x_j) \Psi(x_i) \, dx_i \, dx_j$$

where $U(x_i, x_j)$ is the generic two-particle interaction. Similar representations can be given for three-body and more general many-body interactions.



There are a number of comments we would like to make. Firstly, the creation operators such as $a_s^\dagger$ or $b_s^\dagger$ produce particles in the corresponding states with wave functions $u_s(x)$ or $v_s(x)$. But the creation operator $\Psi^\dagger(x)$ can be thought of as an operator which creates a particle in a state which is localised at the point x itself. Similarly, $\Psi(x)$ can thought of as an operator which destroys particles in ~~the~~ states with precise spatial location.

Consider a state in which one particle is present ~~at~~ on each of N one-particle states $|u_1\rangle, |u_2\rangle \ldots |u_N\rangle$. This state can be written in the form $a_1^\dagger a_2^\dagger \ldots a_N^\dagger |\Phi_0\rangle$, * where $|\Phi_0\rangle$ is the vacuum state, i.e. corresponds to the state where no particles are present at all. As we have already remarked, the ~~surfaces~~ suffices 1...N on the creation operators refer to the particle states, not, of course, to the particle labels, which do not appear at all in the second-quantized formulation. The fact that the original

N-particle theory was a theory of bosons, which stayed symmetric under any permutation of particle labels, corresponds now to the fact that the state given by (*), the commutation properties of the creation operators correspond to the fact that this state is clearly symmetric under the action of state permutations, and, as we have seen, there is a one-to-one correspondence between the particle permutations P and the state permutations $\overline{P}$, although conceptually these two types of permutation are of course utterly distinct. The commutativity of the creation operators is what corresponds, in the second-quantized formulation, to the label permutation symmetry in the original N-particle formulation.

Next we want to stress that we have introduced a quantized field, that is to say, the object $\Psi(x)$, which associates with every point of space and quantum-mechanical operator. It is, if you like, an operator-valued field, or more short a quantum field. But it is very important to notice that this quantum field is merely a mathematical device for reformulating the original particle theory. In no sense have we, ontologically speaking, introduced fields instead of particles. In particular, the harmonic oscillators introduced in the above analysis are a completely formal analogy. They in no way correspond to oscillations of a 'real' field. We shall see later in Chapter 6 below that quantum fields can also be introduced by quantizing

'real' fields. But that is a totally different enterprise from the one we've been engaged in, in the present chapter.

So far we've discussed how you formulate clearly of an assemblage of bosons. Well we now want to explain how this formalism can be adapted to deal with fermions, with the corresponding anti-symmetrisation of their state vectors, and, obeying the Pauli Exclusion Principle, where no more than one particle can occupy any particular quantum state. It's fairly obvious that what we need to do is to introduce a number operator N_s for the s^{th} single particle state, such that its eigenvalues, instead of being 0,1,2... are restricted simply to 0 and 1. This would ensure the operation of the Pauli Exclusion Principle and as we shall see, leads also to the equivalent statement of the anti-symmetrisation property. In fact, it was pointed out originally by Jordan and Wigner in 1926, that all we have to do is replace the commutator bracket in equation ( ) by the anti-commutator bracket, namely

$$[a_s, a_{s'}^\dagger] = \delta_{ss'}, \quad \{a_s, a_{s'}\} = \{a_s^\dagger, a_{s'}^\dagger\} = 0$$

where {} denotes the anti-commutator, so

$$\{A,B\} = AB + BA$$

We now proceed to show that the number operator $N_s = a_s^\dagger a_s$ does in fact have eigenvalues 0 and 1, when we subject these operators to the anti-commutation relations specified above. So we see this formal square

of the operator N_s and, ~~remembering~~ noting that a_s^2 and a_s^{+2} are themselves 0, leads immediately to the result

$$N_s^2 = a_s^+ a_s a_s^+ a_s = a_s^+ (1 - a_s^+ a_s) a_s = a_s^+ a_s = N_s$$

~~resulting in N = N~~ so N_s is an idempotent operator, from which it follows immediately that its eigenvalues are 0 and 1.

To see the connection with anti-symmetrisation, consider again the fermion state $a_1^+ a_2^+ \ldots$ acting on $|\Phi_0\rangle$, ~~and~~ In view of the anti-commutation properties of the creation operators, this state is clearly anti-symmetric under any permutation of the state labels and once again, in view of the one-to-one correspondence between state label permutations and particle label permutations, this now incorporates effectively into the second-quantized theory the requirements of anti-symmetrisation ~~on the~~ under state label permutation we originally imposed on the first-quantized N-particle Schrödinger formalism.

Finally it should be noted that the second-quantized formulation is in fact more general than the first-quantized formulation ~~from~~ for the N-particle problem. For the N-particle problem we have the constraint

$$\sum_s n_s = N$$

but clearly in the second-quantized version this restriction can be relaxed, and leads to the possibility of a theory with a variable number of particles. It is this circumstance which makes the second-quantized formulation so convenient and suitable a vehicle for the description of the ephemeral nature of elementary particles, whose creation and annihilation is such a distinctive feature of ~~elementary particle~~ fundamental physics.

Chapter 5
Relativistic Wave Equations

This will be inserted at a later time.

Chapter 6

Field Quantization

We now want to discuss the quantization of physically real fields, but before we do this, we must enter into some discussion about the nature of a classical field, as disinct from classical particle theory.

Section 6.1

Classical Field Theory

In classical physics it is usual to distinguish particle theories from field theories. The distinction arises in various ways. Firstly there is the contrast between different approaches to the theory of matter. Secondly there is the question of the nature of forces, or interactions between 'pieces' of matter. For field theory such interaction is opposed to 'action-at-a-distance', a doctrine that is sometimes associated with 'pure' particle theories. There is a lot of 'fine structure' in these very crude cut classifications, some of which will emerge in the ensuing discussion.

We start with a rough definition:

A field theory in physics is a theory which associates certain properties with every point of space and time. Examples: Electromagnetic field theory associates electromagnetic forces with space-time points; Eulerian

hydrodynamics associates ~~the~~ a velocity with each space-time point, which represents the velocity of that element of fluid which occupies the indicated position ~~of~~ at the indicated time, and so on.

This definition makes it look as though space-time points are the primary individuals of the theory. The fields are, then, the properties of these spatio-temporal individuals. By contrast, ~~we~~ consider particle theories:

> A particle theory in physics is a theory which attributes certain individuals (the particles) a variety of properties.

These properties will include spatio-temporal locations. ~~Let us capture~~ Notice carefully the different role played by spatio-temporal locations ~~than~~ in ~~earlier~~ our definitions of field theories and particle theories. In field theories, they are individuals associated with properties (the field quantities), in particle theories they are properties associated with individuals (the particles).

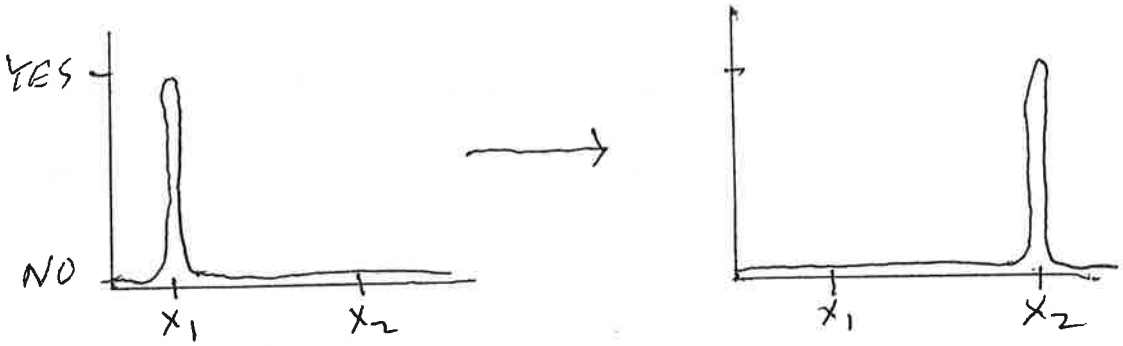
In Chapter 3 we already raised some questions about the problem of individuating space-time points. Since, as we have seen, in field theories the space-time points play the role of the individuals, ~~it~~ the problem of how they get individuated is clearly a very urgent one. Do they possess T.I., or are they individuated by the fields which they ~~already~~ carry? This latter view finds some support in general relativity, but this relies on an application of the identity ~~being assignable~~ of indiscernibles to space-

time points! We shall return to this problem in Chapter 14 below, when we discuss quantum cosmology. For the moment, we merely note that the idea of a persisting individual is of course quite inappropriate to the space-time points. How can a temporal instant persist through time?! Nevertheless, we could regard space-time points as some sort of sui generis particulars. In classical realist absolute space-time theories, these particulars are regarded as ~~being~~ existing, quite independently of material particles which occupy space-time, according to particle theories, and in a pure field ontology, they are the only true particulars there are.

To illustrate the possibilities of the field description, we turn now to discuss how the classical ~~particle~~ mechanics of point particles can be reinterpreted as a field theory. Let us consider the classical particle mechanics of equal point masses. At first sight this looks like a paradigm example of a particle theory. Properties associated with the particles include their positions and momenta, i.e., their location in six-dimensional one-particle ~~face~~ phase space. ~~The~~ For N particles, distinguished by subscripts which play the role of proper names, ~~all its~~ the coordinates and momenta, $\underset{\sim}{q}_i, \underset{\sim}{p}_i, i=1\cdots N$ of all the particles can be assembled into a 6N-dimensional ~~face~~ phase space representing the dynamical state of the complete N-particle system. However, ~~we must~~ it was realised ~~that~~, at least as early as Newton, that particle mechanics can be

represented as a field theory. Instead of a particle as a primary individual, we introduce a dichotomic field, {yes,no} which can take on just two values: 'Yes' indicates that a particle is present at that space-time location, 'no' indicates that a particle is not present. For Newton, 'particle' in this field formulation is identified with the quality ~~of the~~ property of impenatrability, to say "yes" to the question "is there a particle at location ($\underline{r}$,t)?", is just to say that the point ($\underline{r}$,t) has associated with it the quality of impenatrability, an infinite repulsive force field, if you like.

Motion of a particle is represented by the motion of a 'spike' in the dichotomic field, as shown in the following diagram.



Consider for simplicity motion in one dimension, labelled x. The particle at time t_1 is at location x_1, and moves at time t_2 to location x_2. This can be represented as a change in field configuration, ~~with~~ from the spike at x_1 to the spike at x_2. ~~The~~ A property, impenatrability, has been passed on from location x_1 and t_1 to location x_2 and t_2.

Born and Infeld). But now Einstein and relativity had intervened. At first regarded as reactionary in the context of the electromagnetic theory of matter, they gave extra support to the substantial nature of energy, now associated via $E=mc^2$, with inertial mass and ~~following~~ via the Equivalence Principle ~~E.P.~~ with gravitational mass. But general relativity introduced a new field into physics, the metric tensor field, which 'explained' gravitational force. There emerged for Einstein and his collaborators the vision of a Unified Field Theory, in which the electromagnetic field as well as the gravitational ~~can~~ could be given new geometrical significance, and matter, instead of being associated with singularities in this generalised field, points at which the field equations did not apply, ~~was~~ would again be identified with local concentrations of this field. An offshoot of the Einsteinian approach was the geometrodynamics of Wheeler, conceived first as a Unified Field Theory (in the sense of Rainich) but later identified with reamarkable properties of the coupled Einstein-Maxwell equations (theory of geons, etc.).

With this brief survey of classical field theories in mind, we want now to explain quantum field theory.